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Dynamic scaling near the percolation threshold in the diluted Heisenberg chain

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Abstract. Critical spin wave dynamics in the dilute Heisenberg chain near the percolation threshold is treated by two complementary approaches. The first exploits the renormalisation group transformation of parameters under a length scaling achieved by decimation on the equations of motion of the random system; the second obtains from the contributions from chain segments of all possible sizes the average dynamic response using a continuum approach valid for small wavevector k and long percolation correlation length ξ_p. Both approaches yield identical dynamic scaling forms for the dynamic response, with dynamic exponent $z = 2$, and details of the crossover of characteristic frequency between hydrodynamic and critical forms as $k\xi_p$ varies from 0 to ∞. A detailed expression for the scaling function for the dynamic response is also obtained.

1. Introduction

In this paper we investigate the dynamic critical behaviour of the transverse linear response function for a one-dimensional diluted Heisenberg ferromagnet at zero temperature. In such diluted systems (Korenblit and Shender 1978, Kirkpatrick 1979, Stinchcombe 1983a) the dynamics becomes critical near the percolation threshold where the percolation correlation length ξ_p diverges (Essam 1980), and crossover can occur between dynamic behaviours characteristic of wavelengths large or small compared with ξ_p.

The critical dynamic response function is calculated both approximately, using a new dynamic real space renormalisation group (RSRG) technique, and analytically by directly evaluating the dynamic response from clusters of arbitrary size and applying the proper cluster weights.

The RSRG technique exploits the change of parameters (concentration, characteristic frequency, and Green function) occurring under a length scaling of the dilute chain. The transformation is obtained by writing the equation of motion for arbitrary configurations of the random chain and decimating the spin variables for every other site. An account of some aspects of this method has been previously given in Stinchcombe (1983b) and Harris and Stinchcombe (1983). Early versions not dealing fully with arbitrary configurations were developed by Marland and Stinchcombe (see Marland 1977) and Goncalves da Silva and Koiller (1981).

The analytic method depends on considering the dynamics of spins on the isolated pure segments into which the chain separates on dilution, and obtaining the average

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response by superposing the contributions of segments of different lengths with the appropriate probabilities. In the dynamic critical regime a continuum description is possible because both wavelength and ξ_p are large compared with the lattice spacing. This second method has previously been applied to the Glauber dynamics of diluted Ising chains (Harris 1983), where it yields results including those obtained using a different method by Dhar and Barma (1980).

The predictions of the two methods are in complete agreement. The principal result of the paper is that the response function obeys dynamic scaling with a dynamic exponent of 2, but the crossover of characteristic frequency, and a detailed expression for the scaling function of the dynamic response, are also obtained.

The layout of the paper is as follows. The remainder of the present section is spent defining relevant quantities and setting up the model. The renormalisation group calculation is carried out in § 2, and the analytic calculation is performed in §3. The paper is concluded in §4 with a discussion of the results and suggestions for further calculations.

We now proceed to introduce, in turn, basic aspects of the model, the equations of motion and their linearisation, the response function and the dynamic scaling hypothesis.

The system studied is the bond-diluted isotropic Heisenberg chain, in which the exchange coupling J between any pair of nearest-neighbour spins is present with probability p, and absent with probability $1-p$, where p is the bond concentration (see e.g. Stinchcombe 1983b). The Hamiltonian is

$$H = -\sum_l J_l \boldsymbol{S}_l \cdot \boldsymbol{S}_{l+1} - \sum_l \boldsymbol{h}_l \cdot \boldsymbol{S}_l \tag{1.1}$$

where $\boldsymbol{h}_l$ is an arbitrary site-dependent reduced field, and the spins $\boldsymbol{S}_l$ are taken to be classical spins normalised to unity. The random exchange J_l has probability distribution

$$P(J_l) = (1-p)\delta(J_l) + p\delta(J_l - J), \qquad J > 0. \tag{1.2}$$

Diluting a one-dimensional system always breaks it up into finite segments and therefore the threshold p_c for percolation is unity, corresponding to the pure system. For the analytic method (§ 3) we shall need the probability $P(n)$ that a given site of the bond-diluted chain lies in an n-bond segment. This is easily found to be

$$P(n) = (n+1)q^2 p^n \tag{1.3}$$

where

$$q = 1 - p. \tag{1.4}$$

At zero temperature and field the spins in every segment are completely aligned, with the spin directions of different segments uncorrelated. The correlation between a pair of spins separated by r bonds is then the probability that a given pair of such spins lies within the same segment. This yields a correlation function decaying exponentially with r. The reduced correlation length (in units of the lattice spacing) is given by

$$\xi_p = |\ln p|^{-1}. \tag{1.5}$$

This result is also a by-product of the RSRG treatment given in § 2.

We turn now to the dynamics, the main concern of this paper. The Hamiltonian (1.1) leads to the following equation of motion for the spin system:

$$\dot{\boldsymbol{S}}_l = \boldsymbol{S}_l \times (\boldsymbol{h}_l + J_{l-1}\boldsymbol{S}_{l-1} + J_l \boldsymbol{S}_{l+1}). \tag{1.6}$$

In the following, the field $\boldsymbol{h}_l$ is taken as time dependent and transverse to a fixed axis z:

$$\boldsymbol{h}_l(t) = (h_l^{(x)}(t), h_l^{(y)}(t), 0). \tag{1.7}$$

We consider the effect of this transverse field on a (random) chain whose spins are all initially aligned along the z axis. Provided that $\boldsymbol{h}_l(t)$ satisfies certain conditions to be discussed below, the deviation of $\boldsymbol{S}_l$ from the z axis can be made arbitrarily small at all times, and an approximation in which only terms linear in this deviation are retained is appropriate. The resulting equation of motion is

$$-\mathrm{i}\dot{S}_l^+ + J_l(S_l^+ - S_{l+1}^+) + J_{l-1}(S_l^+ - S_{l-1}^+) = h_l^+(t) \tag{1.8}$$

where S^+ and h^+ denote the usual combinations

$$S^+ = S^{(x)} + \mathrm{i}S^{(y)}, \qquad h^+ = h^{(x)} + \mathrm{i}h^{(y)}. \tag{1.9}$$

Taking the time Fourier transform of (1.8) leads to an equation for the Fourier transformed spins $S_l^+(\omega)$. These are related to the Fourier transformed field by the Green function $G_{ll'}(\omega)$ defined as follows:

$$S_l^+(\omega) = -\sum_{l'} G_{ll'}(\omega) h_{l'}^+(\omega). \tag{1.10}$$

With this definition, $G_{ll'}(\omega)$ satisfies the inhomogeneous equation

$$(\omega - J_l - J_{l-1})G_{ll'}(\omega) + J_l G_{l+1\,l'}(\omega) + J_{l-1}G_{l-1\,l'}(\omega) = \delta_{ll'} \tag{1.11}$$

and the imaginary part of the averaged Green function gives the transverse linear response function (generalised susceptibility) for the system:

$$\chi^{(T)}(k, \omega, p) = -\operatorname{Im} \lim_{\varepsilon \to 0} G(k, \omega + \mathrm{i}\varepsilon, p). \tag{1.12}$$

$G(k, \omega, p)$ is here the configuration-averaged Fourier transformed Green function

$$G(k, \omega, p) = \sum_m \mathrm{e}^{-\mathrm{i}km} \langle G_{ll+m}(\omega) \rangle_c. \tag{1.13}$$

$\langle \ldots \rangle_c$ denotes the average over all possible configurations of bonds

$$\langle \ldots \rangle_c \equiv \int \prod_l (P(J_l)\, \mathrm{d}J_l) \ldots \tag{1.14}$$

and in (1.12), (1.13), k is a reduced wavevector (measured in units of the inverse lattice spacing).

In the paper we investigate the dynamic response $\chi^{(T)}(k, \omega, p)$ in the region in which it becomes critical. For the present one-dimensional isotropic Heisenberg system with ferromagnetic interactions, this criticality is at low dilution q (which makes ξ_p large) and small frequency ω and wavevector k. Dynamic scaling arguments (Halperin and Hohenberg 1969) then suggest the following form for the response function:

$$\chi^{(T)}(k, \omega, p) = k^{-(2-\eta^{(T)}+z)} F(k\xi_p, \omega/k^z). \tag{1.15}$$

If the response (1.15) is characterised by some typical frequency ω (e.g. the centre of gravity of one or of a series of peaks in $\chi^{(T)}$, or as defined in Halperin and Hohenberg (1969)), from (1.15) this characteristic frequency will have the homogeneous scaling form

$$\omega = k^z f(k\xi_p). \tag{1.16}$$

Equations (1.15), (1.16) are expected to hold at criticality where k, ω, $1/\xi_p$ all vanish, but with ratios $k\xi_p$, ω/k^z arbitrary.

As indicated earlier, the following sections justify the forms (1.15), (1.16) and yield expressions for the exponents $\eta^{(T)}$, z and the scaling function $F(a, b)$ in (1.15) and in particular its asymptotic forms and those of the scaling function $f(a)$ in (1.16).

The present section is concluded with a brief discussion of the validity of the linear response theory. We note first of all that if a constant field h is applied perpendicular to the spin direction of a particular segment, the spins will precess around the field so that their deviation from their initial direction is no longer small and the linear response theory breaks down. If instead a perpendicular field of amplitude h and frequency ω is applied, and ω is much larger than h, the spin deviation can be shown to be of order h/ω so for finite frequency linear response will be valid for h made arbitrarily small. At criticality, then, h is required to vanish and the ratio h/ω needs to be arbitrarily small, for the response of the system to be linear in the perturbing field. Finally we remark that the transverse linear response for our original system in which a transverse field of form (1.7) is applied, but where the segment magnetisations are randomly oriented, is easily shown to be

$$\bar{\chi}^T(k, \omega, p) = \tfrac{1}{3}\chi^T(k, \omega, p) \tag{1.17}$$

(by averaging the components of the generalised susceptibility tensor).

The next section gives the treatment of the dynamic response and characteristic frequency by the RSRG method.

2. RSRG length scaling approach to dynamics of dilute chain

The basic idea of the RSRG approach (Niemeyer and Van Leeuwen 1976) is to exploit the change of parameters generated by a dilatation of the original lattice. When this dilatation is a scaling of lengths by an integer factor b, it can be achieved by removing ('decimating') spins on a sublattice. In the following we decimate every other spin of the chain, to achieve a length scale change by factor $b = 2$.

The method just introduced is well known in static thermodynamic (Kadanoff and Houghton 1975, Barber 1975) or disorder (Young and Stinchcombe 1975, 1976a, b) problems, where the scale change is carried out subject to the preservation of the Boltzmann factor (for thermodynamic problems) or the basic geometric probabilities (for disorder problems) for the variables that remain. In the case of dynamic problems the decimation method is applied to the equations of motion, preserving the dynamic relationships between the remaining variables. These variables are the S_l^+ of equation (1.8), or, equivalently, the Green functions $G_{ll'}$ of equation (1.11).

The transformation of the Green function equation (1.11), for the disordered chain, is achieved by using similar equations for the Green functions $G_{l\pm1,l'}$ to eliminate

them from equation (1.11), thus arriving at

$$\left(\frac{J_l^2}{D_{l+1}}+\frac{J_{l-1}^2}{D_{l-1}}-D_l\right)G_{ll'}+\frac{J_lJ_{l+1}}{D_{l+1}}G_{l+2\,l'}+\frac{J_{l-2}J_{l-1}}{D_{l-1}}G_{l-2\,l'}$$
$$=\delta_{ll'}+\frac{J_l}{D_{l+1}}\delta_{l+1\,l'}+\frac{J_{l-1}}{D_{l-1}}\delta_{l-1\,l'} \tag{2.1}$$

where $D_l \equiv J_l + J_{l-1} - \omega$. This equation can be regarded as the equation analogous to (1.11) for a system of twice the lattice spacing.

To make the correspondence between the two equations explicit we rewrite them in terms of the reduced variables defined by

$$\Omega_l \equiv \omega/J_l, \tag{2.2}$$

$$g_{ll'} \equiv \omega G_{ll'}. \tag{2.3}$$

The original equation (1.11) then takes the form

$$(1-\Omega_l^{-1}-\Omega_{l-1}^{-1})g_{ll'}+\Omega_l^{-1}g_{l+1\,l'}+\Omega_{l-1}^{-1}g_{l-1\,l'}=\delta_{ll'}. \tag{2.4}$$

In this equation the coefficients of the g's on the left-hand side add up to unity. Introducing the reduced variables into (2.1), and dividing by a factor c_l to make the coefficients on the left-hand side again add to unity, the equation for the scaled system becomes

$$[(\Omega_l^{-2}d_{l+1}^{-1}+\Omega_{l-1}^{-2}d_{l-1}^{-1}-d_l)c_l^{-1}]g_{ll'}+(\Omega_l^{-1}\Omega_{l+1}^{-1}d_{l+1}^{-1}c_l^{-1})g_{l+2\,l'}+(\Omega_{l-2}^{-1}\Omega_{l-1}^{-1}c_l^{-1}d_{l-1}^{-1})g_{l-2\,l'}$$
$$=c_l^{-1}(\delta_{ll'}+\Omega_l^{-1}d_{l+1}^{-1}\delta_{l+1\,l'}+\Omega_{l-1}^{-1}d_{l-1}^{-1}\delta_{l-1\,l'}) \tag{2.5}$$

where $d_l \equiv \Omega_l^{-1}+\Omega_{l-1}^{-1}-1$.

The variables in this equation can now be interpreted as the values to which those in (2.4) (for the original system) transform when the system is dilated by the scale factor $b=2$.

For simplicity we first consider just the rescaled values of the variable Ω_l. These will allow us to obtain the scaling of the characteristic frequency, and hence the scaling form (1.16) and the dynamic exponent z. The transformations obtained by also considering the inhomogeneous terms (right-hand sides) in (2.4), (2.5) will be required later to obtain the full dynamic scaling form for the response function.

In (2.5), the coefficient of $g_{l+2\,l'}$ is the rescaled value of the coefficient of $g_{l+1\,l'}$ in (2.4). Thus under the length scaling by $b=2$,

$$\Omega_l \to \Omega_l' = \Omega_l\Omega_{l+1}d_{l+1}c_l \equiv \phi(\Omega_{l-2}, \Omega_{l-1}, \Omega_l, \Omega_{l+1}) \tag{2.6}$$

(and a similar equation holds for Ω_{l-1}').

Because the bond strengths J_l are independent random variables distributed according to the binary distribution (1.2), the variables Ω_l are also independent binary random variables. (2.6) gives the resulting scaled random variables Ω_l' for a general configuration of the random chain. The distribution of the scaled variables Ω_l' is not binary, nor are they independent. However, we may group the outcomes for Ω_l' into two qualitatively different types, those with (i) finite values, (ii) infinite values. The latter are the analogue of the possibility $J_l=0$ (probability $(1-p)$) in (1.2), and their total probability is $(1-p^2)$. Since this corresponds to the scaled value of the coefficient

$(1-p)$ in the first term in (1.2) we have

$$p \to p' = p^2. \tag{2.7}$$

(2.7) is the usual 'percolation' scaling for the bond concentration of a dilute chain, obtained as usual (Young and Stinchcombe 1975, 1976a) from the probability of a connecting bond (non-zero exchange) in the scaled system. The finite outcomes for Ω_i' have values and probabilities determined by the general random scaling relation (2.6) and the distribution (1.2) of the unscaled variables. These finite outcomes for Ω_i' have total weight p', as given by (2.7), and centre of gravity

$$\Omega' = (1-p)^2(3\Omega - \Omega^2) + (1-p)p[(8-9\Omega+2\Omega^2)/(1-\Omega)] + p^2(4\Omega - \Omega^2). \tag{2.8}$$

Equation (2.8) gives the transformed value of the reduced characteristic frequency variable $\Omega \equiv \omega/J$. This transformation is approximate because it neglects the evolution of the distribution from binary form, and also the appearance of correlations. Such complications have been discussed in the scaling treatment of static disorder problems (Stinchcombe and Watson 1976) and it is known that scaling back to a binary form without correlations is sufficient to maintain the essential features of most dilute systems. Equation (2.8) becomes exact in the pure limit $p = 1$.

The decimation process leading to (2.6), (2.7), (2.8) preserves the probability that the nearest neighbours of the new lattice are joined by a non-zero bond (hence (2.7)) and also the phase relationship of the remaining spin variables. Hence the absolute percolation correlation length and the absolute wavelength of any periodic excitation on the (averaged) lattice are preserved, while the reduced correlation length and reduced wavelength (the corresponding absolute length divided by the lattice spacing) are changed by a factor of a half, because of the doubling of the lattice spacing. Denoting the reduced correlation lengths and reduced wavevectors of the original and scaled lattice by ξ_p, ξ_p', and k, k' (cf (1.5), (1.13)) we thus have

$$\xi_p \to \xi_p' = \xi_p/2, \tag{2.9}$$

$$k \to k' = 2k. \tag{2.10}$$

Since ξ_p' has to be the same function of p' that ξ_p is of p, (2.7) and (2.9) together imply the result (1.5) for the correlation length, already derived in § 1 from cluster statistics. The extraction of such functional relationships from scaling equations is only possible if the scaling equations are very simple. More complicated transformations have to be simplified, e.g. by linearisation around fixed points.

In dilute systems, the scale invariance caused by the divergence of ξ_p at the percolation threshold implies the existence of an unstable (percolation) fixed point. For the one-dimensional case, this is the fixed point

$$p^* = 1 \tag{2.11}$$

of (2.7), which is of course also the pure limit. The fixed point of the dynamic relation (2.8) at which the scale invariance associated with (2.11) and with $k^* = 0$ (equation (2.10)) occurs is

$$\Omega^* = 0. \tag{2.12}$$

The neighbourhood of the combined fixed point (2.11), (2.12) is obviously the dynamic critical regime specified above equation (1.15). In this regime (2.7), (2.8) take the

linearised forms

$$q' = \lambda_p q, \qquad \lambda_p = 2, \tag{2.13}$$

$$\Omega' = \lambda_\Omega \Omega, \qquad \lambda_\Omega = 4 \tag{2.14}$$

where q is the dilution variable defined in (1.4) (the difference of p from its fixed point value), q' is its transformed value, and in (2.13), (2.14) only the terms linear in q and Ω have been retained.

From (2.9), (2.10), (2.13), (2.14) the following variables all transform by the length scale factor 2:

$$1/\xi_p, \qquad k, \qquad q^{\nu_p}, \qquad \Omega^{1/z},$$

where

$$\nu_p = \ln 2/\ln \lambda_p = 1, \tag{2.15}$$

$$z = \ln \lambda_\Omega / \ln 2 = 2. \tag{2.16}$$

This implies the static relationship

$$\xi_p \propto q^{-\nu_p} \tag{2.17}$$

(to which (1.5) reduces in the critical regime $q \equiv 1 - p \ll 1$), as well as the following homogeneous form for the characteristic frequency $\omega = \Omega J$ of the dynamics in the critical regime $k \to 0$, $\xi_p \to \infty$:

$$\omega = k^z f(k\xi_p). \tag{2.18}$$

This is the dynamic scaling form (1.16). From (2.16), the dynamic exponent is $z = 2$.

The renormalisation group method also yields the more general scaling result (1.15), for the response function, by extending the procedure to include also the transformation of the Green function $g_{ll'}$. This transformation is obtained, together with the transformation (2.6) of the random reduced variable Ω_l, by interpreting the full equation (2.5) as a scaled form of (2.4). From the outset we now consider only the dynamic critical regime. There, wavelengths are long and the applied field will differ little from site l to site $l \pm 1$. Or, in other words, the right-hand side of (2.5) is asymptotically equivalent to

$$c_l^{-1}(1 + \Omega_l^{-1} d_{l+1}^{-1} + \Omega_{l-1}^{-1} d_{l-1}^{-1}) \delta_{ll'} = \delta_{ll'}. \tag{2.19}$$

The reduction implied by the simple right-hand side of (2.19) is easily obtained by inserting the detailed forms for $d_{l\pm1}$, c_l into the left-hand side.

With the form (2.19) for the right-hand side of (2.5), the comparison with (2.4) thus yields

$$g_{l+1\,l'} \to g'_{l+1\,l'} = g_{l+2\,l'}, \tag{2.20}$$

etc, as well as the transformation (2.6). Thus all the results previously obtained apply, but in addition we see that the size of the reduced Green function $g'_{ll'}$ is unchanged under scaling. Thus averaging, Fourier transforming, and using (1.12) and (2.3), we obtain

$$\Omega' \chi^{(T)}(k', \Omega', q') = \Omega \chi^{(T)}(k, \Omega, q), \tag{2.21}$$

where k', Ω', q' are related to k, Ω, q by (2.10) and the linearised transformations (2.13), (2.14), which like (2.21) apply in the dynamic scaling regime $k \to 0, q \to 0$.

The combined equations (2.21), (2.10), (2.13), (2.14) imply that $\Omega\chi^{(T)}(k, \Omega, q)$ is a homogeneous function of degree zero in the variables $k, q^{\nu_p}, \Omega^{1/z}$. Thus (using also the values of ν_p, z, given previously in (2.15), (2.16))

$$\chi^{(T)}(k, \Omega, q) = k^{-2} F(k/q, \omega/k^2) \tag{2.22}$$

and the result (1.15) follows with

$$\eta^{(T)} = 2, \qquad z = 2. \tag{2.23}$$

Asymptotic forms of the scaling functions f, F in the dynamic scaling results (2.18), (2.22) can be deduced without further detailed calculation by considering restrictions on the limits as their arguments tend to zero or infinity. For example, in (2.18) $f(a)$ must go to a constant as $a \to \infty$, and to infinity like $1/a^2$ as $a \to 0$, because unlike the cases considered by Halperin and Hohenberg (1969) the characteristic frequency should go to a ξ_p-independent limit as $\xi_p \to \infty$ at fixed k, and to a k-independent limit as $k \to 0$ at fixed ξ_p. These limiting behaviours correspond respectively to pure spin wave dynamics and to the dynamics of the largest spin waves excitable on typical finite clusters.

More detailed results for the form of the dynamic scaling function F for the response, and hence for f, can in principle be obtained from the RSRG method by applying the renormalisation group transformations many times, each time picking up the Green function contribution using (2.20), with the transformed variables given by (2.6), and using the scaled distribution ((1.2) with (2.7)) to evaluate the average. This procedure is analogous to that used for obtaining average free energies of random magnets (Young and Stinchcombe 1976b, Jayaprakash *et al* 1978). It is, however, much simpler in the present (one-dimensional) case to use the method, given in § 3, where the average response is obtained by adding together, with appropriate weights, the contributions from individual finite segments of the chain.

3. Analytical treatment

In this section we write down the equation of motion for the Green function of an n-bond segment. Then a continuum approach appropriate to criticality is used to establish the dynamic scaling form (1.15) for $\chi^{(T)}$, and the values of the exponents $\eta^{(T)}$ and z. Finally an explicit calculation is performed to obtain the scaling function $F(a, b)$.

Using (1.11), the Green function $G_{ll'}^{(n)}(\omega)$ for an n-bond Heisenberg chain segment satisfies the following equations of motion: when l is an interior site

$$\omega G_{ll'}^{(n)}(\omega) + J[G_{l+1\,l'}^{(n)}(\omega) + G_{l-1\,l'}^{(n)}(\omega) - 2G_{ll'}^{(n)}(\omega)] = \delta_{ll'}, \qquad 1 \leq l \leq n-1, \tag{3.1}$$

or an end site

$$\begin{aligned} &\omega G_{0l'}^{(n)}(\omega) + J[G_{1l'}^{(n)}(\omega) - G_{0l'}^{(n)}(\omega)] = \delta_{0l'}, \\ &\omega G_{nl'}^{(n)}(\omega) + J[G_{n-1\,l'}^{(n)}(\omega) - G_{nl'}^{(n)}(\omega)] = \delta_{nl'}. \end{aligned} \tag{3.2}$$

The site labels on the chain segment run from 0 to n and $G_{ll'}^{(n)}(\omega)$ is zero if l or l' is outside the range 0 to n. The configuration-averaged Green function for the bond-diluted chain can easily be expressed in terms of the finite chain Green functions.

The result is

$$\langle G_{ll+m}(\omega)\rangle_c = \sum_{n=|m|}^{\infty} q^2(1-q)^n \sum_{l=0}^{n-|m|} G^{(n)}_{ll+|m|}(\omega). \tag{3.3}$$

We note that the configuration-averaged Green function depends only on $|m|$ and is thus translationally invariant, as assumed implicitly in some discussions of § 2. Substituting (3.3) into (1.13) and rearranging yields the result

$$G(k,\omega,p) = \sum_{n=0}^{\infty} q^2(1-q)^n \sum_{m=0}^{n} 2\cos km \sum_{l=0}^{n-m} G^{(n)}_{ll+m}(\omega). \tag{3.4}$$

The difference equations (3.1) can be solved subject to the boundary conditions (3.2) and an explicit expression for $G^{(n)}_{ll}(\omega)$ then follows. On substituting this into (3.4), the sums over l and m can be evaluated leaving exact expressions for $G(k,\omega,p)$ and $\chi^{(T)}$ in the form of sums over n, which cannot however be evaluated in closed form. Instead of following through this procedure we write down continuum forms of equations (3.1)–(3.4) appropriate at criticality. Dynamic scaling forms for G and $\chi^{(T)}$ then follow naturally without further explicit calculation. Furthermore, we shall see that the continuum approach leads to a closed form expression for the scaling function $F(a,b)$.

We begin by defining a new frequency variable ν and scaled lengths x, y and L as follows:

$$\nu^2 \equiv \Omega = \omega/J, \qquad y \equiv l/\xi_p, \qquad x \equiv m/\xi_p, \qquad L \equiv n/\xi_p. \tag{3.5}$$

We shall also require the scaled variables $\tilde{\nu}$ and $\tilde{k}$ which are defined as

$$\tilde{\nu} \equiv \nu\xi_p, \qquad \tilde{k} \equiv k\xi_p. \tag{3.6}$$

At criticality the variables $1/\xi_p$, ν and k all tend to zero. Then the spacing between adjacent values of y becomes infinitesimal

$$\Delta y = 1/\xi_p \to dy \tag{3.7}$$

and, from (1.5), (1.4),

$$1/\xi_p \sim q. \tag{3.8}$$

Because of (3.7), the difference equations (3.1) and (3.2) go over to the following differential equations:

$$\tilde{\nu}^2\tilde{G}^{(L)}(y,y';\tilde{\nu}) + \frac{\partial^2}{\partial y^2}\tilde{G}^{(L)}(y,y';\tilde{\nu}) = J^{-1}\delta(y-y'), \qquad 0<y<L, \tag{3.9}$$

$$\frac{\partial\tilde{G}^{(L)}}{\partial y}(0,y';\tilde{\nu}) = \frac{\partial\tilde{G}^{(L)}}{\partial y}(L,y';\tilde{\nu}) = 0. \tag{3.10}$$

The scaled Green function $\tilde{G}^{(L)}(y,y';\tilde{\nu})$ is given by

$$\tilde{G}^{(L)}(y,y';\tilde{\nu}) = \xi_p^{-1}G^{(n)}_{ll'}(\omega). \tag{3.11}$$

Taking the continuum limit in (3.4), using (3.7) and corresponding equations for x and L, and using also (3.8) we obtain

$$G(k,\omega,p) = \xi_p^2\int_0^{\infty} dL\, e^{-L}\int_0^{L} dx\, 2\cos\tilde{k}x\int_0^{L-x} dy\,\tilde{G}^{(L)}(y,y+x;\tilde{\nu}) = k^{-2}g(\tilde{k},\tilde{\omega}) \tag{3.12}$$

where

$$\tilde{\omega} = \omega \xi_p^2, \tag{3.13}$$

$$g(\tilde{k}, \tilde{\omega}) = \tilde{k}^2 \int_0^\infty dL\, e^{-L} \int_0^L dx\, 2 \cos \tilde{k}x \int_0^{L-x} dy\, \tilde{G}^{(L)}(y, y+x; \tilde{\nu}). \tag{3.14}$$

Thus $G(k, \omega, p)$ and hence $\chi^{(T)}$ obey the dynamic scaling form (1.15) at criticality, with the following exponents:

$$z = 2, \qquad \eta^{(T)} = 2. \tag{3.15}$$

The scaling function $F(a, b)$ in (1.15) is given by

$$F(a, b) = -\lim_{\tilde{\varepsilon} \to 0} \operatorname{Im} g(a, ba^2 + i\tilde{\varepsilon}). \tag{3.16}$$

The present section is concluded with the evaluation of a closed form expression for $F(a, b)$, and its interpretation. The solution of (3.9) subject to the boundary conditions (3.10) is straightforward and yields the result

$$\tilde{G}^{(L)}(y, y'; \tilde{\nu}) = \frac{\cos(L - |y - y'|)\tilde{\nu} + \cos(L - y - y')\tilde{\nu}}{2J\tilde{\nu} \sin L\tilde{\nu}}. \tag{3.17}$$

Substituting (3.17) into (3.14) and evaluating the integrals over y and x leads to the following expression for $g(\tilde{k}, \tilde{\omega})$:

$$g(\tilde{k}, \tilde{\omega}) = \frac{\tilde{k}^2}{J(\tilde{\nu}^2 - \tilde{k}^2)} + \frac{2\tilde{k}^4}{\tilde{\nu}(\tilde{\nu}^2 - \tilde{k}^2)^2} \int_0^\infty dL\, \frac{e^{-L}(\cos \tilde{\nu}L - \cos \tilde{k}L)}{J \sin \tilde{\nu}L}. \tag{3.18}$$

The zero-frequency pole in (3.18) present for all $\tilde{k}$ is due to the breakdown of linear response at zero frequency for all values of the wavevector k for a diluted system (individual finite segments see a net static field). Performing the operation on the right-hand side of (3.16) removes the pathologies in the integral over L in (3.18) and yields the following closed form result for the scaling function $F(a, b)$:

$$F(a, b) = \frac{\pi \coth[(\pi/2a)\sqrt{J/b}][1 + \cos(\pi\sqrt{J/b})]}{a^2 b(b/J - 1)^2[\cosh[(\pi/a)\sqrt{J/b}] + \cos(\pi\sqrt{J/b})]} \tag{3.19}$$

where $a = k\xi_p$, $b = \omega k^{-2}$. The result (3.19) is a smooth function (not a series of δ-function peaks) because the critical dynamic response comes from the very long segments (typically of length $\sim \xi_p$) for which the smoothing implicit in the continuum limit is appropriate.

We can investigate the asymptotic forms of $F(a, b)$ in the hydrodynamic ($k\xi_p \to 0$) and critical ($k\xi_p \to \infty$) limits. These are easily obtained from (3.19) and are as follows

$$F(a, b) \underset{a \to 0}{\sim} \frac{2\pi(1 + \cos \pi\sqrt{J/b})}{a^2 b(b/J - 1)^2} \exp\left[-\frac{\pi}{a} \left(\frac{J}{b} \right)^{1/2} \right], \tag{3.20}$$

$$\lim_{a \to \infty} F(a, b) = \pi \delta(b - J). \tag{3.21}$$

The response function (3.20) for the hydrodynamic limit ($k\xi_p \to 0$) vanishes (because of the exponential) unless b varies in such a way as to keep ba^2 of order J. In other words, the characteristic frequency in the hydrodynamic regime is

$$\omega \sim J/\xi_p^2. \tag{3.22}$$

In the critical limit $k\xi_p \to \infty$ the response is, from (3.21), the spin wave peak at the characteristic frequency

$$\omega \sim Jk^2. \tag{3.23}$$

(3.22) and (3.23) are the same asymptotic behaviours obtained for the characteristic frequency by the crossover scaling arguments at the end of § 2, and correspond respectively to the following statements: $f(a)$ goes to J/a^2 for $a \to 0$ and to the constant value J for $a \to \infty$. The constant value, J, is the pure spin wave stiffness (Harris and Kirkpatrick 1977, Korenblit and Shender 1978, Ziman 1979).

4. Discusssion

Dynamic scaling forms for the response function have been obtained both by recursive rescaling (RSRG) techniques and by an analytic method making explicit use of cluster statistics. These two approaches yield the same scaling form (1.15) and the same values of the dynamic and static exponents z, $\eta^{(T)}$ (compare (2.23) and (3.15)). Dynamic scaling forms for the characteristic frequency are given directly by the RSRG method or, in general, via the scaling form for the response function. The asymptotic behaviours given by the two methods are again in agreement and have simple physical interpretations. A detailed expression for the principal scaling function $F(a, b)$ (in (1.15)) has been obtained by the analytic method, and may be useful for comparison with experiment. $F(a, b)$ is also in principle available from the RSRG technique: both methods (RSRG and analytic) developed here seem able to give complete treatments of the problem, though some results follow more easily from one than from the other. The complementary point of view (length scaling aspects of the random system, dynamics and statistics of finite clusters) that they provide seems valuable.

The methods have only been applied to a very simple problem: the transverse dynamics of the bond-diluted Heisenberg ferromagnet at zero temperature. Though its behaviour is much richer than might at first be expected, many generalisations are obviously desirable. For experimental comparisons, site dilution rather than bond dilution is of more interest. This generalisation modifies the discussion of the response function since the site dilution affects the inhomogeneous terms, as well as the exchange couplings, in the Green function equations. This problem, along with others now mentioned, is left for future discussion.

Another development desirable for contact with experiment is the generalisation to the antiferromagnet, since localised antiferromagnets are far commoner than ferromagnets. This generalisation is again in principle possible: a two-sublattice description, leading to slightly different continuum equations from those for the ferromagnetic case, is involved in the analytic method; the rescaling treatment is most easily accomplished by using an initial $b = 2$ decimation to transform the system to a ferromagnet, and thereafter proceeding as in this paper. We expect for the dilute antiferromagnet Heisenberg chain qualitatively similar behaviour to that found above (dynamic scaling forms for response function and characteristic frequency, the latter crossing over between hydrodynamic and propagating behaviour as $k\xi_p$ ranges from 0 to ∞). However, details such as the scaling function for the response will certainly be different, and so will be the dynamic exponent z since, while Heisenberg antiferromagnets and ferromagnets are in the same static universality class, their critical dynamics differs because they do not share the same conserved quantities (Hohenberg

and Halperin 1977). Indeed, it is obvious from the discussion in this paper that for the bond-diluted chain in its critical region ($k \to 0, \xi_p \to \infty$) the dynamic exponent is the same as for the pure case, so for the bond-diluted antiferromagnetic chain we expect $z = 1$ in its critical regime. A proper treatment of this case is nevertheless very desirable, so that detailed comparison can be made with the inelastic neutron scattering results of Boucher *et al* (1978) and Endoh *et al* (1979) on the diluted quasi-one-dimensional Heisenberg antiferromagnet $(CD_3)_4N(Mn_pCu_{1-p})Cl_3$. These are the only experiments so far known on the critical dynamics of dilute Heisenberg chains.

Most important is the generalisation to higher dimensions. This allows the possibility of greater contact with experiment, and introduces the following new features: the percolation threshold is at $p_c < 1$, and while for $p < p_c$ only the finite cluster response occurs (as in one dimension), for $p > p_c$ both finite and infinite cluster response appears; also, the infinite cluster response is different from the pure response, for $p < 1$, and cluster shape now matters. The crossover between different dynamic behaviours is much richer than in the one-dimensional case we have treated. While some steps of the applications used here are obviously limited to one dimension, some of the more central ideas do generalise. The application of the RSRG rescaling method to the problem of critical dynamics in the neighbourhood of the percolation threshold in higher-dimensional systems has already been briefly described elsewhere (Harris and Stinchcombe 1983).

A further important feature ignored in the present analysis but briefly discussed elsewhere (Stinchcombe 1983b) for the dilute Heisenberg chain is the nature of the recursive scaling (e.g. equation (2.8)) of dynamic quantities in the rescaling approach. Such nonlinear recursion relations include the possibility of chaotic as well as periodic orbit behaviour, related respectively (in the scaling of the chain) to pure and finite cluster dynamics. It was not necessary to discuss this feature in the present paper because only the dynamic scaling regime was considered in detail, where linearised recursion relations apply.

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